

# Analogy between turbulence and quantum gravity: beyond Kolmogorov's 1941 theory

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## Abstract

Simple arguments based on the general properties of quantum fluctuations have been recently shown to imply that quantum fluctuations of spacetime obey the same scaling laws of the velocity fluctuations in a homogeneous incompressible turbulent flow, as described by Kolmogorov 1941 (K41) scaling theory. Less noted, however, is the fact that this analogy rules out the possibility of a fractal quantum spacetime, in contradiction with growing evidence in quantum gravity research. In this Note, we show that the notion of a fractal quantum spacetime can be restored by extending the analogy between turbulence and quantum gravity beyond the realm of K41 theory. In particular, it is shown that compatibility of a fractal quantum-space time with the recent Horava-Lifshitz scenario for quantum gravity, implies singular quantum wavefunctions. Finally, we propose an operational procedure, based on Extended Self-Similarity techniques, to inspect the (multi)-scaling properties of quantum gravitational fluctuations.

*Keywords: Fractal space-time, fluid turbulence, scaling laws*

## 1 Introduction

Quantum Gravity (QG) and Fluid Turbulence (FT) stand out as two major unsolved challenges in modern theoretical physics. This is largely due to the fact that, despite their distinct physical nature, they are both characterized by strong non-linearities, which hinder the development of a fully-fledged theory. As a result, any formal analogy between these two fields is potentially of interest, since it may permit to put ideas and techniques developed in one field, to the benefit of the other. For instance, the recent analogies between

Navier-Stokes and Einstein equations shed some hope that concepts and methods from the celebrated AdS/CFT duality [1, 2], can be used to attack fluid turbulence through the formulation of a weakly-interacting gravitational dual. In this paper, we shall discuss a potential contribution in the reverse direction, namely the possibility of using hierarchical models developed in the framework of the phenomenology of fluid turbulence, to the potential benefit of QG research. Analogies between fluctuating hydrodynamics and quantum gravity have been noticed since long. Visually, the QG-FT analogy was first brought up by Wheeler's poignant description of the so-called quantum foam [3], i.e. metric fluctuations which become comparable in size with the metric background at the Planck scale. Subsequently, using simple arguments of general relativity and quantum theory, Padmanabhan was able to point out operational limits in measuring the position of a particle to a better accuracy than the Planck length [4].

Recently, the analogy has been tightened, by showing that the quantum fluctuation of spacetime obey the same scaling laws of velocity fluctuations of turbulent flows, as described by Kolmogorov's 1941 theory [5]. The argument goes as follows [6, 7]. Consider a wavepacket traveling from a source S to a mirror point M a distance  $l = |M - S|$  apart. Classically, the distance  $l$  is measured simply as  $l = ct_r/2$ ,  $c$  being the speed of light and  $t_r$  the return time at which the signal transmitted at time  $t = 0$  is received back by the source S. Due to Heisenberg's principle, the distance  $l$  is subject to an uncertainty  $\delta l$ , which cannot be made smaller than the wavepacket width at time  $t_r$ ,  $w(t_r)$ . The spread of a quantum wavepacket propagating in free space is given by  $w^2(t) = w^2(0) + \frac{D_q^2 t^2}{w^2(0)}$ , where  $D_q = \hbar/m$  is the quantum diffusivity. This expression has a minimum at  $w_{min} = \sqrt{2 D_q t_r}$ , and consequently,

$$(\delta l)^2 \geq \frac{D_q l}{c} = l \lambda_c, \quad (1)$$

where  $\lambda_c = \hbar/mc$  is the Compton wavelength and numerical factors have been taken to unity for simplicity. Differently restated,

$$\frac{\delta l}{\lambda_c} \geq \left(\frac{l}{\lambda_c}\right)^{1/2} \quad (2)$$

at super-atomic scales  $l > \lambda_c$ . Note that the fluctuation scales non-analytically with  $l$ , i.e. their gradient  $g(l) \sim \frac{\delta l}{l}$  would diverge like  $l^{-1/2}$  in the limit  $l \rightarrow 0$ . Of course, such limit makes no physical sense, because UV cutoffs must be taken into account, which is where gravity takes the stage. The gravitational bound reads simply

$$\delta l > \lambda_s \quad (3)$$

where  $\lambda_s = Gm/c^2$  is the Schwarzschild length. This condition ensures that the source S has not melted down into a black hole by the time the signal is back. Note that at these scales the fluctuations are even more singular, as they don't even go to zero in the limit  $l \rightarrow 0$ , a manifestation of gravitational singularities.

Neither this limit, however, is the relevant one at Planckian scales, which live exactly at the geometrical mean between the Compton and the Schwartzschild scales,  $\lambda_p^2 = \frac{G\hbar}{c^3} = \lambda_s \lambda_c$ . Here, the relevant scaling is the given by the combination of quantum and gravitational constraints, that is  $(\delta l)^3 > \lambda_c \lambda_s l = \lambda_p^2 l$ , or, differently restated,

$$\frac{\delta l}{\lambda_p} > \left(\frac{l}{\lambda_p}\right)^{1/3} \quad (4)$$

Thus, at the Planck scale we still find a singular behaviour of  $\delta l$ , with an intermediate exponent (1/3) between the Schwartzschild (0) and Compton (1/2) regimes. Note that in all cases, the gradient  $g(l)$  is singular, with exponent  $-1$ ,  $-2/3$  and  $-1/2$ , in the Schwartzschild, Planck and Compton regimes, respectively. Interestingly, 1/3 is exactly the exponent predicted by Kolmogorov's 1941 theory (K41) of homogeneous, incompressible turbulence [8, 9]. More precisely, K41 predicts that the velocity fluctuations in a turbulent flow, scale like  $\delta v(l) = v_k (l/l_k)^{1/3}$ , where  $l_k$  is the Kolmogorov dissipative length and  $v_k \equiv \delta v(l_k)$ . The analogy is apparent, upon the identification  $\delta l \leftrightarrow \delta v$  and  $\lambda_p \leftrightarrow l_k$ .

For fluid turbulence, the 1/3 exponent results from a specific assumption on the physical nature of dissipative processes, namely the scale invariance of the dissipation rate (energy dissipated per unit volume and unit time), that is:

$$\epsilon(l) \equiv \frac{\delta v^2(l)}{\tau(l)} \sim \frac{\delta v^3(l)}{l} = \text{const.} \quad (5)$$

In [9], it was further noted that the exponent 1/3 implies that the set where energy is dissipated is space-filling, ruling out fractals. Following upon the exact analogy, a similar statement would also apply to the metric fluctuations, thereby casting questions on the various theories of fractal quantum spacetime [10, 11]. Therefore, either the simple derivation of the 1/3 law in quantum gravity is inaccurate, or the fractal theory of quantum spacetime must be revisited. In view of the mounting evidence of non-integral and scale-dependent effective dimensions of spacetime [12], here we pursue the former alternative. In the following, we shall show that the notion of a fractal quantum spacetime can be reconciled with the turbulence analogy, provided the analogy is extended to well-known generalizations of the K41 theory. Most interestingly, such generalizations are shown to be compatible with recent statistical field theory formulations of QG, the so-called Horava-Lifshitz picture [13], as well as the Dynamic Triangulation (DT) approach to numerical quantum gravity. The K41 picture is recovered in the IR limit of a smooth 4-dimensional spacetime. The QG-FT analogy can be "continued" towards the UV scales, on condition of replacing K41 with its well-know generalizations (Kolmogorov's 1961, K61 for short).

## 1.1 Kolmogorov K41 theory, cascade models and multi-fractals

Before addressing these generalized scenarios, let us briefly revisit the way fractal sets (fail to) emerge within the standard K41 theory of turbulence. To this

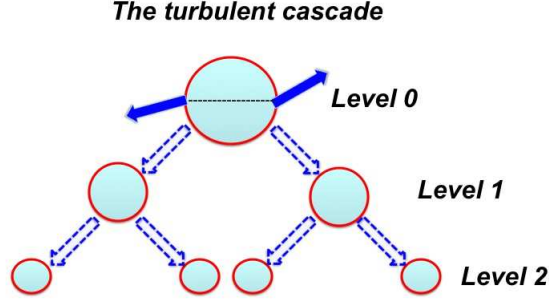


Figure 1: The Richardson cascade of turbulent eddies. The fluctuating quantity is the velocity fluctuation across the eddy (solid arrows). For simplicity, this is shown for the mother eddy only.

purpose, let us remind the central notion of energy cascade in fluid turbulence. This refers to the picture of a turbulent flow as a collection of coherent excitations (eddies). Under the effect of non-linear interactions, large eddies break down in smaller eddies, which in turn further break down in smaller eddies, and so on down the line, until the Kolmogorov length is reached, below which dissipation takes over, thereby terminating the energy cascade. To be noted that energy cascades from large to small scales virtually unchanged, all dissipation taking place at and below the Kolmogorov scale. The assumption that the dissipation rate be scale-invariant actually implies that dissipation is a homogeneous process, filling up the entire space occupied by the turbulent fluid. A very convenient modeling framework to quantify this idea, if only on phenomenological grounds, is provided by the so-called hierarchical cascade models [14]. Within this model, the energy cascade can be cartooned as follows. A mother eddy at scale  $l_0$  breaks down into 2 eddies of scale  $l_1 = rl_0$ , where  $r < 1$  is an arbitrary fragmentation factor. The daughter eddies, in turn, break down into "niece" eddies of size  $l_2 = rl_1 = r^2l_0$  and so on down the line, till the  $N$ -th eddy meets the Kolmogorov scale,  $l_0r^N = l_k$ , thereby terminating the cascade. The velocity fluctuations associated with an eddy of  $n$ -the generation are assumed to scale like  $\frac{\delta v_n}{\delta v_0} = (l_n/l_0)^h$ , where  $v_0$  is a **typical fluid velocity at scale  $l_0$**  and  $h$  is an unknown exponent at this point. Imposing scale-invariance of the energy flux, i.e.  $\delta v^3/l = \text{const}$ , yields  $3h = 1$ , singling out  $h = 1/3$  as a scaling exponent. Central to this result is the space-filling character of the cascade, i.e. each and every fragmented eddy fragments entirely into further daughter eddies. A qualitatively different picture emerges by assuming that only a volumetric fraction  $\beta < 1$  of the daughter eddies remains active for further fragmentation.

The dissipation rate at the  $n$ -th generation is now given by  $\epsilon_n = \beta^n \frac{\delta v^3}{l_n}$ . Let us now posit  $\beta^n = (\frac{l}{l_0})^d$ , which defines the parameter  $d = D - D_f$  as the defect dimension, i.e. the co-dimension of the set where the cascade takes place,  $D_f$  being its fractal dimension in an embedding space of dimension  $D$ . By its very definition,  $d = \frac{\log \beta}{\log r}$ , from which it is seen that the space-filling scenario,  $\beta = 1$ , yields  $d = 0$ , i.e. no fractal set. By imposing again scale invariance of  $\epsilon_n$ , we now obtain  $3h - 1 + d = 0$ , i.e.

$$h = \frac{(1-d)}{3} \quad (6)$$

This highlights a one-to-one connection between the fractal dimension of the cascade space and the scaling exponent of the velocity fluctuations which live on it. In passing, we note that the most violent fluctuations take place on sets with large co-dimensions  $d$ , i.e. small fractal dimension  $D_f$ . In fluid turbulence, the most singular structures are typically credited for being filaments of fractal dimension  $D_f = 1$ ,  $d = 2$  and  $h = -1/3$ .

It is well known that real-life turbulent flows do *not* obey the relation (6), i.e. they are not described by a single fractal, of whatever dimension. Experimentally, this is revealed by inspecting the so-called structure functions of order  $p$ :

$$S_p(l) = \langle \delta v^p(l) \rangle = (\delta v_0)^p \left(\frac{l}{l_0}\right)^{\zeta_p} \quad (7)$$

where brackets stand for ensemble averaging over turbulent realizations and  $\zeta_p$  are the corresponding scaling exponents of order  $p$ . By construction, high  $p$ 's probe rare and highly energetic events (bursts). **Should turbulence be described by a single fractal, as per eq. (6), one would observe a linear dependence of the scaling exponents on  $p$ , namely  $\zeta_p = hp$ .** This is contradicted by experimental evidence, which shows instead a sub-linear dependence of the form

$$\zeta_p = hp + \eta_p \quad (8)$$

where  $\eta_p < 0$  describe the so-called "intermittency" corrections. Intermittency refers to the fact that, as suggested by visual experience, turbulence is all but a homogeneous process. Quite on the contrary, dissipation typically comes through spotty bursts and gusts, which stand in stark contrast with the notion of a scale-independent, space-filling, homogeneous dissipation rate. This observation motivated the development of Kolmogorov's 1961 theory [15]. By using the definitions (5) and (7), we readily obtain  $S_p(l) = \langle \epsilon^{p/3} l^{p/3} \rangle \sim l^{p/3} l^{\tau_p/3}$ , where  $\tau_p$  are the scaling exponents of the dissipation field. Comparison with (8), shows that  $\eta_p \equiv \tau_p/3$ , i.e. deviations of the scaling exponents  $\zeta_p$  from linear behaviour can be ascribed to the fluctuations of the dissipation rate. A powerful notion to account for intermittency, if not explain it, is the concept of multifractal, whereby turbulent dissipation takes place on a sequence of fractals (multifractal), each with its own fractal co-dimension  $d(h)$ , varying within a continuous range  $h_1 \leq h \leq h_2$ . Although multifractals do not provide an explanation for intermittency, they set nonetheless a powerful mathematical stage for

a geometrical theory of turbulence [16]. Here, we conclude this brief excursus by noting that, at level of cascade modeling, the notion of multifractal is readily incorporated by promoting the volumetric fraction  $\beta$  from a mere parameter to a random distribution (random beta model).

## 1.2 Quantum gravity inverse cascade

Back to quantum gravity, the question is whether the QG-FT analogy survives the extensions of Kolmogorov's theory, and, if so, how does one accommodate the notions of intermittency, multifractals and associated cascade models. In the sequel, we shall show that the analogy can indeed be taken to this extended territory, provided quantum spacetime fluctuations are treated like a fractional brownian motion, in line with recent theories of anisotropic spacetime and quantum gravity at a critical Lifshitz point [13]. Borrowing the cascade language for quantum gravity, one postulates an *inverse* cascade (UV to IR), whereby two small mother eddies of size  $l_0$  coalesce into a single daughter eddy of size  $l_1 = l_0/r$  ( $r < 1$ ) and so on, *up* the line. The formalism applies all the same, in reverse, with the obvious duality  $r \rightarrow 1/r$  and  $\beta \rightarrow 1/\beta$ , which leaves  $d$  unchanged. Thus, if metric fluctuations live in a fractal spacetime of dimension  $D_f = D - d$ , their scaling exponent is  $h = (1 - d)/3$ , violating the simple scaling relation eq. (4). Since many evidences are now accumulating for a fractal spacetime, with spectral dimension ranging from about 2 at high-energy, up to the full 4 dimensions at low energy [12], it makes sense to revisit the main procedure leading to the  $1/3$  exponent. To this purpose, let us assume that the signal propagation obeys a generalized Brownian motion of the form  $dl = c_z dt^{1/z}$ , where  $z$  is the dynamical exponent at a critical Lifshitz point. The case  $z = 1$  recovers standard advection (smooth manifold), and  $z = 2$  to standard diffusion (non-differentiable manifold). The recent Horava-Lifshitz (HL) proposal of anisotropic spacetime fluctuations (Lifshitz critical point), predicts a spacetime spectral dimension  $D_s = 1 + D/z$ , in  $D$  spatial dimensions. More precisely, the HL theory predicts  $z = 3$ , i.e.  $D_s = 2$  in the UV range, up to  $z = 1$ , i.e.  $D_s = 4$  at large scales. In our language,  $D/z \equiv D - d$ , i.e.  $d = D(1 - 1/z)$ , so that  $z = 3$  and  $z = 1$  yield  $d = 2$  and  $d = 0$  respectively in  $D = 3$ , corresponding to spectral dimension  $D_s = 2$  and  $D_s = 4$  respectively, in close match with numerical quantum gravity simulations based on causal dynamic triangulations (CDT) of spacetime [17, 18]. To be noted that the corresponding exponents run from  $h = 1/3$  (IR) to  $h = (1 - 2)/3 = -1/3$ , pointing to a very singular UV behavior of the metric fluctuations. These fluctuations are wilder than turbulent velocity fluctuations, but less violent than the fluctuations of the dissipation field  $\epsilon(l) \sim l^{-2/3}$ . In this sense, the QG-FT analogy still holds, although more in relation to flow dissipation than the flow velocity itself. Assuming that the minimum spread of the wavepacket still obeys the Schroedinger scaling  $\delta_{min}^2 \sim Dt$ , the Schwartzschild constraint  $\delta l > \lambda_S$ , as combined with a HL return time  $t_r \sim l^z$ , would yield  $(\delta l)^3 \sim l^z$ , that is  $\delta l \sim l^{z/3}$ , i.e.  $h = z/3$ . With  $z = 1$ , this returns the familiar exponent  $h = 1/3$ , while  $z = 3$  yields  $h = 1$ , in stark contrast with the previous finding  $h = -1/3$ . Clearly, some

assumptions need to be revisited. In particular, in a fractal spacetime, there is no reason to believe that the spatial spreading of the wavepacket should obey the same diffusion-like relation as in smooth spacetime. Assuming that in the HL spacetime the minimum wavepacket spread obeys a scaling relation of the form  $\delta_{min}^2 \sim t_r^{\alpha(z)}$ , combination with  $t_r \sim l^z$ , would give  $h = \alpha(z)z/3$ . Clearly, in the limit  $z \rightarrow 1$  we require  $\alpha \rightarrow 1$ , so as to recover the standard K41 scenario. Compatibility with the relation  $h = (1 - d)/3$  and the identity  $d = D(1 - 1/z)$ , yields a spreading exponent

$$\alpha(z) = \frac{1}{z} - D\left(\frac{1}{z} - \frac{1}{z^2}\right) \quad (9)$$

This relation is reported in Figure 2 as a function of  $z$  for the case  $D = 3$ . First, we note that indeed  $\alpha$  goes to the unit value for  $z = 1$ , as it should. Second, we observe that the spreading exponent becomes negative for  $z > z^*(D) = \frac{1}{1-1/D}$ , namely  $z^* = 3/2$  for  $D = 3$ . Third, the minimum exponent is attained at  $z = z_{min}(D) = \frac{2}{1-1/D} = 2z^*$ , its value being  $\alpha_{min} = -\frac{(D-1)^2}{4D}$ . Amazingly, for  $D = 3$ , the minimum value,  $\alpha_{min} = -1/3$  is attained precisely at the HL value  $z = 3$ . Finally, and most importantly, we observe that a negative spreading exponent implies the development of a finite-time singularity in the quantum wavefunction carrying the propagating signal. Singular wavefunctions are known since long in quantum mechanics [19], however to the best of this author knowledge, they have not been discussed before in the framework of quantum gravity.

### 1.3 Extended Self Similarity for quantum gravity

The existence of a continuous range of spectral exponents is conducive to the idea of QG multifractals. In order to detect them, it appears natural to define QG structure functions and associated scaling exponents:

$$G_p(l) = \langle \left(\frac{\delta l}{\lambda_p}\right)^p \rangle \sim \left(\frac{l}{\lambda_p}\right)^{\gamma_p} \quad (10)$$

These could be measured in QG simulations, possibly using the techniques of extended self-similar similarity, to be detailed shortly [20, 21]. In the sequel, we provide an operational procedure to accomplish this task. Let us write the volume change from scale  $l$  to  $l + \delta l$  as

$$\delta V(l) \equiv V(l + \delta l) - V(l) = v_1 l^2 \delta l + v_2 l (\delta l)^2 + v_3 (\delta l)^3 \quad (11)$$

where  $v_k$ ,  $k = 1, 3$  are  $O(1)$  geometry-dependent constants. **For a cube**  $v_1 = v_2 = 3$  **and**  $v_3 = 1$  **and for a sphere the same sequence is pre-factored by**  $4\pi/3$ . For a generic tetrahedron in a CDT realization of spacetime,  $v_k$  should be taken as random numbers, whose statistics contains all the non-perturbative physics beyond simple scaling, if any.

For a generic scaling law of the form  $\frac{\delta l}{\lambda_p} \sim \left(\frac{l}{\lambda_p}\right)^h$ , one readily computes  $y \equiv \frac{\delta V(l)}{V_p} = v_1 x^{2+h} + v_2 x^{1+2h} + v_3 x^{3h}$  where  $h < 1$  is the scaling exponent,

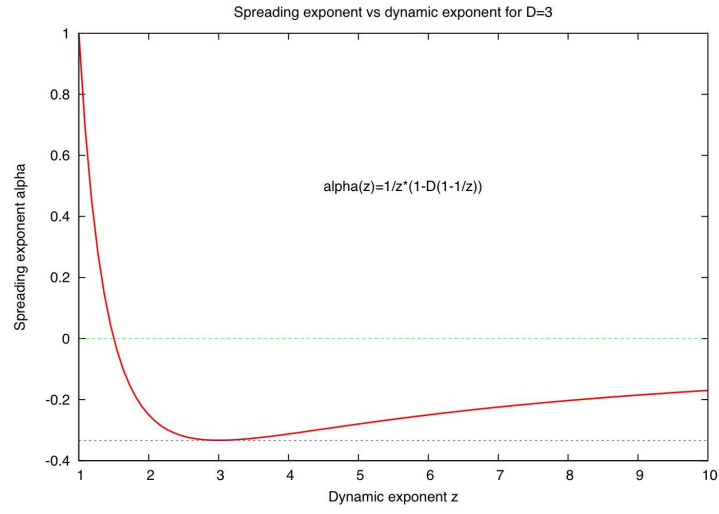


Figure 2: The spreading exponent of the quantum wavefunction,  $\alpha$ , as a function of the dynamic exponent  $z$  for the case of three spatial dimensions  $D = 3$ . To be noted that, for  $z > 3/2$ , the spreading exponent becomes negative, pointing to the development of a finite-time singularity in the quantum wavefunction carrying the propagating signal.



$V_p = \lambda_p^3$  is the Planck volume, and we have set  $x \equiv l/\lambda_p$ . If the moments of  $v_k$  remain finite at any order, then  $\gamma_p = (2+h)p$ , otherwise, multifractal behaviour is observed. For  $x \gg 1$ , the first term on the rhs leads the series, hence the scaling exponent of volume fluctuations is simply  $2+h$ . One could obtain  $\gamma_p$  by log-plotting  $G_p$  versus  $x$ , using the data from a CDT simulation. In actual practice, however, the scaling relation  $y \sim x^{2+h}$  usually holds only in a restricted interval  $1 \ll x \ll \Lambda \equiv L/\lambda_p$ , usually too small to allow statistical accuracy (present day CDT simulations feature  $\Lambda \sim 10^2$ ). While waiting for computer advances, one must turn to alternative tools. One which turned out to be pretty useful to unravel the scaling properties of turbulent flows is Extended Self Similarity (ESS). ESS maintains a generalized scaling law of the form

$$y \sim \psi(x)^{2+h} \quad (12)$$

where  $\psi(x)$  is a universal function of  $x$ , which reduces to  $x$  only in the limit  $x \gg 1$  (one can think of it as of a generalized space coordinate  $x' = \psi(x)$ , probing a larger set of scales). The ESS scaling (12) then holds on a broader range than the native scaling  $y \sim x^{2+h}$ . Unfortunately,  $\psi(x)$  is generally not known a-priori, and consequently ESS cannot be used to deduce the exponent by log-plotting  $y$  vs  $\psi(x)$  from numerical data. However, *relative* scaling functions can be used to bypass the problem. It is indeed clear that under ESS conditions, the following relative scaling between two generic structure functions of order  $p$  and  $q$ , holds

$$G_p \sim (G_q)^{\gamma_p/\gamma_q} \quad (13)$$

where  $\gamma_p/\gamma_q$  reduces to  $p/q$  for the case of simple scaling (no multifractal). In case the scaling exponent is analytically known for some reference value  $q^*$ , relative scaling of  $G_p$  versus  $G_{q^*}$ , uniquely delivers  $\gamma_p$ . In fluid turbulence  $q^* = 3$  and  $\gamma_3 = 1$ . We are not aware of a corresponding QG analogue. The advantages of ESS analysis are clear: i) no knowledge of  $\psi(x)$  is required, ii) the extraction of  $\gamma_p$  from numerical data can rely on a broader range of values. These have proven especially valuable in low-resolution experiments in fluid turbulence, where standard scaling would not hold on a sufficiently broad range of wavenumbers. The same advantage is expected apply to the statistical data from CDT simulations. The procedure described above is unambiguous, hence fully operational once a set of CDT data is available

## 2 Conclusions

Based on the quantitative analogy between the scaling exponent of quantum spacetime fluctuations and fluid turbulence within Kolmogorov K41 theory, we conclude that the common exponent  $h = 1/3$  rules out the possibility of a fractal quantum spacetime. In this Note, we have shown that such a possibility can be restored by moving to the generalized Kolmogorov 1961 theory and drawing a parallel with a critical Horava-Lifshitz QG scenario, whereby space and time would fluctuate with different exponents. Within such generalized scenario, QG

fluctuations look more akin to the fluctuations of the dissipation rate rather than to velocity fluctuations and imply the development of finite-time singular quantum wavefunctions. Finally, an operational procedure to measure the (multi)-scaling properties of quantum gravitational fluctuations, based on Extended Self Similarity techniques, has been suggested.

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